Key Objectives:

1) Know how to determine the seismic velocity structure from refraction data for multiple layers.

2) Understand why low velocity layers cause problems for the seismic refraction technique and how to determine if a low velocity layer exists.

3) Understand the challenges associated with thin layers.

4) Qualitatively be able to determine if a layer is dipping, and if it is dipping be able to determine by how much.

1. Multiple Layers

1.1 Travel Times

Last time we derived the travel time equation for a simple case with only two layers. But, you’re probably wondering how useful that is in the Earth. As we will show later, this can be very useful for studying major boundary layers such as the Moho. But, true, velocities in the Earth change with depth as well as laterally. Here, let’s just look at the case where velocity changes with depth.

We can make use of thin isotropic layers to represent (i.e., to approximate) any vertical profile of seismic velocities, e.g., here’s a linear gradient approximated by discrete steps (i.e., discrete constant velocity layers):

Thus, we can always pick a layer thickness appropriate for the accuracy of which we want approximate structure. How do we deal with these multiple layers? Simple we can just trace our seismic ray downwards using Snell’s Law at each interface:
Similar to the two-layer case we also get critically refracted arrivals at each interface:
The angle nomenclature for the above diagram is as follows:

**Critical Angles**

| \( \theta_{1,2} \) | Critical angle between layers 1 & 2 |
| \( \theta_{2,3} \) | Critical angle between layers 2 & 3 |

**Non-Critical Angles (Defined by Snell’s Law at each interface)**

| \( \theta_{1,3} \) | angle of refraction in layer 1 for ray critically refracted in layer 3 |
| \( \theta_{1,4} \) | angle of refraction in layer 1 for ray critically refracted in layer 4 |
| \( \theta_{2,4} \) | angle of refraction in layer 2 for ray critically refracted in layer 4 |

To determine the travel times we proceed just as we did in the two-layer case. That is, just add up the travel times for each ray segment, do some algebra and trig...

What we find is that the travel time for the critically refracted ray from the top of layer \( n \) is given by:

\[
T_n = t_{n-1} + \frac{x}{v_n}
\]  
(Eqn 1)

Note that this is still the equation of a straight line with a slope = \( 1/v_n \) and intercept given by,

\[
t_{n-1} = \sum_{i=1}^{n-1} \frac{2h_i \cos \theta_{i,n}}{v_i}
\]  
(Eqn 2)

Remember, this only works for the case where: \( v_1 < v_2 < v_3 < v_4 \) … That is, these equations only work when you have increasing velocity with depth.

**How do these equations work?**

- **Direct Arrival (n = 1)** – the direct arrival is given by \( n=1 \), or \( T_1 \). If we plug \( n=1 \) into Eqn 1:

\[
T_1 = t_0 + \frac{x}{v_1}
\]

So, we can immediately see that as expected the slope of the direct arrival is: \( 1/v_1 \). If we plug \( n=1 \) into Eqn 2:

\[
t_0 = \sum_{i=1}^{0} \frac{2h_i \cos \theta_{i,1}}{v_i} = 0
\]
- **First Critically Refracted Arrival** \((n = 2)\) – this is the refraction along the \(v_1-v_2\) boundary. As a reminder this is the same situation that we derived the travel times for in the last lecture. So, plugging \(n = 2\) into Eqn 1:

\[
T_2 = t_{2-1} + \frac{x}{v_2} = t_1 + \frac{x}{v_2}.
\]

So, as expected the slope of the line is given by \(1/v_2\). Now, plugging \(n = 2\) into Eqn 2:

\[
t_1 = \sum_{i=1}^{1} \frac{2h_i \cos \theta_{i,2}}{v_i} = \frac{2h_1 \cos \theta_{1,2}}{v_1}
\]

And, we note that the intercept is exactly as we derived before. Again, for clarification this is the arrival we are talking about:

![Diagram of first critically refracted arrival](image)

- **Second Critically Refracted Arrival** \((n = 3)\) – this is the refraction along the \(v_2-v_3\) boundary. Plugging \(n = 3\) into Eqns. 1 and 2:

\[
T_3 = t_2 + \frac{x}{v_3}
\]

\[
t_2 = \sum_{i=1}^{2} \frac{2h_i \cos \theta_{i,3}}{v_i} = \frac{2h_1 \cos \theta_{1,3}}{v_1} + \frac{2h_2 \cos \theta_{2,3}}{v_2}
\]

So, we can see that now the slope is equal to \(1/v_3\). The intercept is a little more complicated now, but there is enough information to now calculate the thickness of the 2nd layer (i.e., \(h_2\)). As a reminder this is the arrival we are now talking about:
So, you get the idea. What do the travel time curves look like if you put them all on one plot?

So, as you can see that measuring the first arrivals when there are multiple layers can be tricky as there may be many slopes. But, by measuring these slopes and intercepts we can determine the velocities and thicknesses of the layers!
1.2 Critical Distances

As in the two-layer case we need to know the critical distances. Recall from our last lecture the two-layer case:

\[
\tan \theta_c = \frac{b}{h} \\
\Rightarrow b = h \tan \theta_c \\
\Rightarrow X_c = 2b = 2h \tan \theta_c
\]

If we have multiple layers finding the critical distances is the same, we just need to add up the horizontal segments of the up- and down-going rays. The figure for the 2nd critical refraction is drawn on the next page.

The critical distance is then given by:

\[
X_c = 2(d_1 + d_2) \\
\Rightarrow X_c = 2(h_1 \tan \theta_{1,3} + h_2 \tan \theta_{2,3})
\]
2. Problem Situations

We showed above how to find velocities and layer thickness for multiple layers, but there are some special situations where we run into problems.

2.1. Low Velocity Layer

A problem may arise if we have a low velocity layer, e.g., if \( v_2 < v_1 \). So, what happens to our ray paths when we encounter a lower velocity? Recall Snell’s law:

\[
\theta_2 = \sin^{-1} \left( \frac{v_2}{v_1} \sin \theta_1 \right),
\]

which gives us the following three cases:

\[
\begin{align*}
v_2 &= v_1 \Rightarrow \theta_2 &= \theta_1 \\
v_2 &= v_1 \Rightarrow \theta_2 &> \theta_1 \\
v_2 &= v_1 \Rightarrow \theta_2 &< \theta_1
\end{align*}
\]
So when we have a low velocity layer the ray gets bent backward and no critically refracted ray can be initiated:

![Diagram showing travel time and distance relationship](image)

So, what does the travel time plot look like?
So, the slope of the travel time curve is \(1/v_3\) and not \(1/v_2\). What this means is that from looking at the travel time curves we might interpret the structure as being due to a 2-layer structure and not a 3-layer structure. Hence, if we then tried to get thickness of the layers we would erroneously come up with a structure that looks like this:

So, how can we detect a low velocity layer using the refraction technique? The trick is to look at the critical distance. Consider the critical distances for the *measured structure* vs. the *true structure*:
If we assume just a two-layer model then the critical distance \((X_c)\) is larger than what we actually observe. Again, looking at a travel-time curve:
So, if you have a dense enough array of receivers you may be able to determine if there is a low velocity layer.

2.2. Thin Layer

Another problem may arise when there exists a very thin layer, such as is shown below with the 2\textsuperscript{nd} layer (thickness = \(h_2\)).
Here, the arrival from the critically diffracted ray along the $v_2$-$v_3$ boundary could always arrive ahead of the arrival from the critically diffracted ray along the $v_1$-$v_2$ boundary. This is because the velocity in layer 3 is higher than the velocity in layer 2. In this case, the travel time curves might look as follows:

![Travel Time Curve Diagram](image)

Note that the travel-time curve for the 1st critically refracted arrival never shows up as the 1st arrival. Hence it may be hidden and may result in an overestimation in thickness of the next layer.

3. Dipping Interface

Up to now everything we did was for flat lying layers. Obviously the Earth is not so simple. So what if we have a dipping interface. The next figure sets up the problem:
Here, we have the following nomenclature:

- \( h_d \): Depth to dipping layer perpendicular to dipping layer (on the downdip side)
- \( h_u \): Depth to dipping layer perpendicular to dipping layer (on the updip side)
- \( z_d \): Depth to dipping layer perpendicular to surface (downdip side)
- \( z_u \): Depth to dipping layer perpendicular to surface (updip side)
- \( \alpha \): Dip angle of dipping layer

Again, finding the travel time for the refracted ray is just a matter of a little algebra and trigonometry. The solution is:

\[
\text{Travel Time (down dip)} = T_d = 2h_d \cos \theta_c \frac{\cos \alpha}{v_1} + x \frac{\sin \theta_c}{v_1}
\]

Again, note that this is just the equation of a straight line. But the slope is: \( \frac{\sin(\theta_c + \alpha)}{v_1} \). So, the slope of depends on both the critical angle and the dip angle of the interface, but also goes as \( 1/v_1 \) and not \( 1/v_2 \) as we saw in the case of flat lying layers.

Now what would happen if we reversed the location of the source and receivers, such that now we were shooting the ray up dip. The solution for this case is:

\[
\text{Travel Time (up dip)} = T_u = 2h_u \cos \theta_c \frac{\cos \alpha}{v_1} + x \frac{\sin \theta_c - \alpha}{v_1}
\]
Notice that both the slope and intercept are different! So, it makes a difference whether you are shooting seismic rays up- or down-dip.

Here’s what the travel time curves would look like:

\[ b = \frac{2h_u \cos \theta_C}{v_1} \]
\[ b = \frac{2h_d \cos \theta_C}{v_1} \]

So, if we wanted to find out if a layer was dipping what we need to do is take two measurements, where we reverse the source and receiver for the second measurement. So how do we determine the thickness of the layer underneath the source and receiver?

Let’s write the slope of the down-dip travel time as:

\[ m_d = \frac{\sin(\theta_C + \alpha)}{v_1} = \frac{1}{v_d} \]

So, \( m_d \) just stands for the slope of the down-dip shot, and we define this as equal to \( 1/v_d \) – the apparent velocity for the down-dip shot.

Then we can rewrite this equation as:

\[ m_d = \frac{\sin(\theta_C + \alpha)}{v_1} = \frac{1}{v_d} \]

\[ \Rightarrow \sin(\theta_C + \alpha) = \frac{v_1}{v_d} \]

\[ \Rightarrow \theta_C + \alpha = \sin^{-1}\left(\frac{v_1}{v_d}\right) \]
If we also define the slope of the up-dip shot as being equal to $1/v_u$, then we can rewrite the slope for the up-dip shot in the same way. Hence we get two equations with two unknowns:

$$\theta_c + \alpha = \sin^{-1}\left(\frac{v_1}{v_d}\right),$$

$$\theta_c - \alpha = \sin^{-1}\left(\frac{v_1}{v_u}\right),$$

where the unknowns are $\theta_c$ (the critical angle) and $\alpha$ (the dip angle). To solve for $\theta_c$ we can simply add the two equations:

$$\theta_c + \alpha + \left(\theta_c - \alpha\right) = \sin^{-1}\left(\frac{v_1}{v_d}\right) + \sin^{-1}\left(\frac{v_1}{v_u}\right),$$

$$\Rightarrow 2\theta_c = \sin^{-1}\left(\frac{v_1}{v_d}\right) + \sin^{-1}\left(\frac{v_1}{v_u}\right),$$

$$\Rightarrow \theta_c = \frac{1}{2}\left(\sin^{-1}\left(\frac{v_1}{v_d}\right) + \sin^{-1}\left(\frac{v_1}{v_u}\right)\right)$$

To solve for $\alpha$ we just subtract the two equations:

$$\theta_c + \alpha - \left(\theta_c - \alpha\right) = \sin^{-1}\left(\frac{v_1}{v_d}\right) - \sin^{-1}\left(\frac{v_1}{v_u}\right),$$

$$\Rightarrow 2\alpha = \sin^{-1}\left(\frac{v_1}{v_d}\right) - \sin^{-1}\left(\frac{v_1}{v_u}\right),$$

$$\Rightarrow \alpha = \frac{1}{2}\left(\sin^{-1}\left(\frac{v_1}{v_d}\right) - \sin^{-1}\left(\frac{v_1}{v_u}\right)\right)$$

Then we are able to solve for $v_2$ using Snell’s Law:
\[
\sin \theta_c = \frac{v_1}{v_2}
\]

\[
\Rightarrow v_2 = \frac{v_1}{\sin \theta_c}
\]

And finally, the thicknesses \( h_d \) and \( h_u \) can now be solved for from the equations for the intercepts.

4.0 Examples

Here is an example worked out to show how some of this works.

4.1 Three layers, non-dipping interface

We have collected seismic data on a receiver line that is 300 km long. From our seismograms we have measured the travel-times of the arrivals and plotted them all on the following figure as travel-time vs. distance.

Our first step is to interpret the above figure. It may look confusing at first, but with refraction data we are looking for groupings of arrivals that fall along straight lines, so the easiest thing to do is to just lay a straight edge on the plot and see what lines up. In this example I see three arrivals. So, I next fit these data as best as I can with straight lines as shown in the next figure.
I interpret this is follows:

1. **Red line** – direct seismic arrival (since it passes through the origin and has the shallowest slope).
2. **Green line** – first critically refracted arrival (steeper slope than the direct arrival, indicating a faster velocity, but not the steepest slope).
3. **Purple line** – 2nd critically refracted arrival (steepest slope, indicating the fastest seismic arrival and the deepest layer).

The next step is to calculate the slopes of each of the lines and to determine the seismic velocities of each layer.

**Direct arrival:**

This is easy from this plot:

\[
slope \text{ of direct arrival} = \frac{150 \text{ s}}{300 \text{ km}} = 0.5 \text{ s/km}
\]

Because the velocity is the inverse of the slope, we can determine the velocity of this layer as:

\[
V_1 = \frac{1}{0.5 \text{ s/km}} = 2.0 \text{ km/s}
\]
First critically refracted arrival:

From the plot we can measure the slope of the 1\textsuperscript{st} critically refracted arrival.

\[
\text{slope first critically refracted arrival} = \frac{104 \text{ s}}{260 \text{ km}} = 0.4 \text{ s/km}
\]

\[
V_2 = \frac{1}{0.4 \text{ s/km}} = 2.5 \text{ km/s}
\]
2nd critically refracted arrival:

Similar to the last case we measure the slope:

\[
\text{slope second critically refracted arrival} = \frac{62.25 \text{ s}}{218 \text{ km}} = 0.2856 \text{ s/km}
\]

\[
V_3 = \frac{1}{0.2856 \text{ s/km}} = 3.5 \text{ km/s}
\]

At this point we are doing good. We have determined that our measurements support the existence of 3 layers and we have determined the seismic velocities of each of these layers. Next we need to determine the thicknesses of each of these layers. The next figure summarizes what we know so far.
Angles from Snell’s Law

In order to determine the thickness of these layers we need to first calculate some angles. Angles for the second critically refracted arrival are drawn in the above figure. We can determine these angles using Snell’s Law.

First we calculate the critical angle $\theta_{2,3}$.

$$\frac{\sin(\theta_{2,3})}{v_2} = \frac{\sin(90^\circ)}{v_3}$$

$$\Rightarrow \theta_{2,3} = \sin^{-1}\left(\frac{v_2}{v_3}\right)$$

$$\Rightarrow \theta_{2,3} = \sin^{-1}\left(\frac{2.5 \text{ km/s}}{3.5 \text{ km/s}}\right) = 45.5847^\circ$$

Now we can use Snell’s Law again to calculate $\theta_{1,3}$:

$$\frac{\sin(\theta_{1,3})}{v_1} = \frac{\sin(\theta_{2,3})}{v_2}$$

$$\Rightarrow \theta_{1,3} = \sin^{-1}\left(\frac{v_1}{v_2} \sin(\theta_{2,3})\right)$$

$$\Rightarrow \theta_{1,3} = \sin^{-1}\left(\frac{2.0 \text{ km/s}}{2.5 \text{ km/s}} \sin(45.5847^\circ)\right) = 34.8499^\circ$$
We also need to know the critical angle between the top two layers $\theta_{1,2}$:

$$\frac{\sin(\theta_{1,2})}{v_1} = \frac{\sin(90^\circ)}{v_2}$$

$$\Rightarrow \theta_{1,2} = \sin^{-1}\left(\frac{v_1}{v_2}\right)$$

$$\Rightarrow \theta_{1,2} = \sin^{-1}\left(\frac{2.0 \text{ km/s}}{2.5 \text{ km/s}}\right) = 53.13^\circ$$

**Intercept Times**

Next we need to determine the intercept times for each of the critically refracted arrivals.

Measuring the intercept time for the first critically refracted arrival:

We measure an intercept time $t_1 = 9$ s.
Now we can measure the intercept time for the second critically refracted arrival:

We measure an intercept time $t_2 = 29.14 \text{ s}$.

**Layer Thicknesses**

Now we have everything we need to calculate the layer thicknesses. From the equations given above:

$$t_1 = \frac{2h_1 \cos(\theta_{1,2})}{v_1}$$

$$\Rightarrow h_1 = \frac{t_1 v_1}{2 \cos(\theta_{1,2})}$$

$$\Rightarrow h_1 = \frac{(9 \text{ s})(2.0 \text{ km/s})}{2 \cos(53.13^\circ)} = 15 \text{ km}$$

And,

$$t_2 = \frac{2h_1 \cos(\theta_{1,3})}{v_1} + \frac{2h_2 \cos(\theta_{2,3})}{v_2}$$
\[
\Rightarrow \frac{2h_2 \cos(\theta_{2,3})}{v_2} = t_2 - \frac{2h_1 \cos(\theta_{1,3})}{v_1}
\]

\[
\Rightarrow h_2 = \frac{v_2}{2 \cos(\theta_{2,3})} \left( t_2 - \frac{2h_1 \cos(\theta_{1,3})}{v_1} \right)
\]

\[
\Rightarrow h_2 = \frac{(2.5 \text{ km/s})}{2 \cos(45.5847^\circ)} \left( 29.14 \text{ s} - \frac{2(15 \text{ km}) \cos(34.8499\degree)}{(2.0 \text{ km/s})} \right) = 30 \text{ km}
\]

From the information we recorded we cannot determine the thickness of the third layer. Thus, we have determined the seismic velocities of three layers and the thicknesses of the top two layers.

**Critical Distances**

Next we should check our critical distances to ensure there are no hidden layers.

The critical distance for the first refracted arrival is given by:

\[
X_{c,1} = 2h_1 \tan(\theta_{1,2})
\]

\[
\Rightarrow X_{c,1} = 2(15 \text{ km}) \tan(53.13\degree) = 40 \text{ km}
\]

And,

\[
X_{c,2} = 2(h_1 \tan(\theta_{1,3}) + h_2 \tan(\theta_{2,3}))
\]

\[
X_{c,2} = 2((15 \text{ km}) \tan(34.8499\degree) + (30 \text{ km}) \tan(45.5847\degree)) = 82 \text{ km}
\]

We can compare these critical distances with our observations as shown in the next figure. Here we can see that the critical distances we predicted above match well with the earliest distances that we observe our two critically refracted arrivals. Thus we have gained some confidence in our model and conclude that a significant low velocity layer does not exist.
Our final model looks as follows:

\[ v_1 = 2.0 \text{ km/s} \quad h_1 = 15 \text{ km} \]
\[ v_2 = 2.5 \text{ km/s} \quad h_2 = 30 \text{ km} \]
\[ v_3 = 3.5 \text{ km/s} \quad h_3 = \text{unknown} \]